Similarly, $2\int \cos^2\frac{1}{2}zd\varphi=-\psi\cos x+2\tan^{-1}\left(\frac{\sin\frac{1}{2}(y+x)\tan\frac{1}{2}\psi}{\sin\frac{1}{2}(y-x)}\right)$.

$$\int\cos zd\psi=\int(\cos x\cos y+\sin x\sin y\cos\psi)d\psi=\psi\cos x\cos y+\sin x\sin y\sin\psi.$$

When $z=x+y$, $\theta=\varphi=0$, $\psi=\pi$; when $z=x-y$, $\theta=\psi=0$, $\theta=\pi$; when $z=y-x$, $\varphi=\psi=0$, $\theta=\pi$.

$$\therefore\ \triangle=\tfrac{1}{2}\pi r^2\int_0^{\frac{1}{2}\pi}\left[\int_0^x(3+\cos x\cos y-\cos x-\cos y)dy\right.$$

$$\left.+\int_x^{\frac{1}{2}\pi}(3+\cos x\cos y-\cos x-\cos y)dy\right]dx.$$

$$\therefore\ \triangle=\tfrac{1}{2}\pi r^2\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(3+\cos x\cos y-\cos x-\cos y)dxdy$$

$$=\tfrac{1}{4}\pi r^2\int_0^{\frac{1}{2}\pi}(3\pi+2\cos x-\pi\cos x-2)dx=\tfrac{1}{8}\pi r^2(3\pi^2+4-4\pi)$$

129. Proposed by J. K. ELLWOOD, Principal of Colfax School, Pittsburg, Pa.

A and B play with two dice, A throwing. If he throws 7 or 11, he wins; if he throws 3, or two aces, or two sixes, B wins. But if he throws 4, 5, 6, 8, 9, or 10, he continues throwing to duplicate this throw, in which event he wins; if in throwing, however, he throws 7, B wins. What is the expectancy of each? [This is the regulation "crap" game, B being banker.]

Solution by G. B. M. ZERR, A..M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The chance of throwing 7 or 11 is $\frac{2}{9}$; the chance of throwing 2, 3, or 12 is $\frac{1}{9}$; the chance of throwing 4, 5, 6, 8, 9, or 10 is $\frac{2}{3}$. If A throws 4 the first throw the chance of winning the second throw is $\frac{1}{12}\cdot\frac{2}{3}$; of winning the third throw is $\frac{1}{12}\cdot\frac{2}{3},[1-(\frac{1}{12}+\frac{1}{6})]=\frac{1}{12}\cdot\frac{2}{3}\cdot\frac{3}{4}$.

$\therefore$ A's chance of winning on 4 is $\frac{2}{9}+\frac{1}{12}\cdot\frac{2}{3}[1+\frac{3}{4}+(\frac{3}{4})^2+(\frac{3}{4})^3+\ldots.]=\frac{4}{9}$.

A's chance of winning on 5 is $\frac{2}{9}+\frac{1}{9}\cdot\frac{2}{3}[1+\frac{13}{18}+(\frac{13}{18})^2+(\frac{13}{18})^3+\ldots]=\frac{22}{45}$.

A's chance of winning on 6 is $\frac{2}{9}+\frac{5}{36}\cdot\frac{2}{3}[1+\frac{25}{36}+(\frac{25}{36})^2+(\frac{25}{36})^3+\ldots.]=\frac{52}{99}$.

A's chance of winning on 8, 9, or 10 is the same as for 6, 5, or 4.

$\therefore$ A's chance$=\frac{1}{3}(\frac{4}{9}+\frac{22}{45}+\frac{52}{99})=\frac{722}{1485}$; B's chance$=1-\frac{722}{1485}=\frac{763}{1485}$.

If the wager is given their expectation follows at once.

Also solved, with different results, by *W. W. LANDIS*.

MISCELLANEOUS.

128. Proposed by J. E. SANDERS, Hackney, O.

The sides of a trapezium are $a=29$, $b=32$, $c=40$, and $d=36$. If c is opposite a, and the diagonals equal, what is the length of either diagonal?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $AD=a=29$, $BC=q=32$, $CD=c=40$, $DA=d=36$, $AC=BD=x$, $\angle DOA=\theta$. Project AD, BC, AC on BD.

$\therefore$ $x=d\cos ADB+b\cos CBD+x\cos\theta$. Multiply through by $2x$ and write, $-\cos(ADB+CAD)$ for $\cos\theta$.

$\therefore$ $2x^2=2dx\cos ADB+2bx\cos CBK-2x^2\cos(ADB+CAK)$; $2dx\cos AKB=d^2+x^2-a^2$; $2bx\cos CBK=d^2+x^2-c^2=2dx\cos CAK$. Substituting,

$$2x^2=d^2+x^2-a^2+d^2+x^2-c^2-1/2d^2(d^2+x^2-a^2)(d^2+x^2-c^2)$$
$$+1/2d^2\sqrt{\{[4d^2x^2-(d^2+x^2-a^2)^2][4d^2x^2-(d^2+x^2-c^2)^2]\}}.$$

Reducing and collecting,

$$2x^6-(a^2+b^2+c^2+d^2)x^4+[a^2(b^2-2c^2+d^2)+b^2(c^2-2d^2)+c^2d^2]x^2$$
$$+(ac-bd)(ac+bd)(a^2-b^2+c^2-d^2)=0.$$

Restoring numbers, $2x^6-4761x^4+317712x^2+2238016=0$.

$\therefore$ $x=48.07$ nearly.

Also solved by *LON C. WALKER*, *J. SCHEFFER*, and *D. B. NORTHRUP*. Mr. Northrup's result agreed with Professor Zerr's and was obtained by the method of trial and error.

129. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

How high above the surface of the earth must an observer be elevated at the latitude $\phi(=39^\circ\ 19')$, the declination of the sun being $\delta(=23^\circ\ 27')$, in order to see the sun at midnight?

Solution by the PROPOSER.

The sun will be seen at midnight when the tangent drawn from the point to the earth strikes the sun when on the meridian at midnight. Denoting the required height above the earth by h, the radius of the earth by R, the latitude of the place by ϕ, and the declination of the sun by δ, we easily find $\sin(\phi+\delta)=\dfrac{R}{R+h}$, whence $h=\dfrac{R[1-\sin(\phi+\delta)]}{\sin(\phi+\delta)}=\dfrac{2R\sin^2[45-\frac{1}{2}(\phi+\delta)]}{\sin(\phi+\delta)}$.

For $\phi=39^\circ\ 19'$, $\delta=23^\circ\ 27'$, we get $h=495$ miles, nearly.

Also solved, with slightly different results, by *G. B. M. ZERR*, *S. HART WRIGHT*, and *G. W. GREENWOOD*.